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FLUCTUATION ENERGY EQUATIONS FOR TURBULENT FLUID–SOLID FLOWS

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Abstract—Fluctuation energy equations and the associated constitutive relations are derived for turbulent fluid–solid flows. This work is a continuation of previous studies, where the continuity and momentum equations were derived. The framework comprising all these equations is valid for all flow regimes. Closure of the constitutive relations will depend on the nature of the various flow regimes. The phase interaction terms in the fluid and solid fluctuation energy equations differ by the term $\langle m_i\rangle(\{v_i\} - \{u_i\})$. When the phase interaction in the momentum equation, $\langle m_i\rangle$, is dominated by the drag force, the above interaction term in the energy equation results in an enhancement of fluid turbulence by the presence of particles.

Key Words: fluctuation, energy, particles, turbulence, interaction, constitutive

1. INTRODUCTION

Formulation of the governing equations in a two-phase flow has been an ongoing research topic. A number of recent works have treated the two phases separately. Balance laws have been proposed for both phases, with interaction terms linking the two (e.g. Ishii 1975; Drew 1983; Prosperetti & Jones 1984; McTigue *et al.* 1986; Ahmadi 1987; Ma 1987). Utilizing the ensemble average, $\langle\ \rangle$, and the mass-weighted average, $\{\ \}$ (Ahmadi & Shahinpoor 1983), the balance equations for mass and momentum have the following general form:

solid phase,

$$\frac{\partial\langle\rho^{s}\rangle}{\partial t}+\frac{\partial}{\partial x_i}(\langle\rho^{s}\rangle\{u_i\})=0 \qquad [1]$$

and

$$\langle\rho^{s}\rangle\left(\frac{\partial\{u_i\}}{\partial t}+\{u_j\}\frac{\partial\{u_i\}}{\partial x_j}\right)=\langle\rho^{s}\rangle g_i+\langle m_i\rangle+\frac{\partial}{\partial x_j}(\langle c\rangle\{T^{s}_{ji}\}); \qquad [2]$$

and

fluid phase,

$$\frac{\partial\langle\rho^{f}\rangle}{\partial t}+\frac{\partial}{\partial x_i}(\langle\rho^{f}\rangle\{v_i\})=0 \qquad [3]$$

and

$$\langle\rho^{f}\rangle\left(\frac{\partial\{v_i\}}{\partial t}+\{v_j\}\frac{\partial\{v_i\}}{\partial x_j}\right)=\langle\rho^{f}\rangle g_i-\langle m_i\rangle+\frac{\partial}{\partial x_j}(\langle 1-c\rangle\{T^{f}_{ji}\}). \qquad [4]$$

In the above, $\rho^{s}=\rho_s c$ and $\rho^{f}=\rho_f(1-c)$ are, respectively, the partial densities of the solid and fluid phase, c is the solid volume concentration, ρ_s and ρ_f are the solid and fluid material density, respectively; u_i is the solid velocity and v_i is the fluid velocity; T^{s}_{ji} and T^{f}_{ji} are the stresses for the solid and fluid phase, respectively; g_i is gravity; and m_i denotes the phase interaction force acting on the solid by the fluid per unit volume of the mixture.

Other averaging processes, e.g. the time average (Ishii 1975) and volume average (Prosperetti & Jones 1984), also resulted in the same equations as shown above. However, the interpretations of the terms and constitutive relations are different. The governing equations and the associate constitutive relations must be chosen consistently to ensure correct modeling. As discussed in Hwang & Shen (1989), one area of confusion in two-phase flow modeling is that, even with the same ensemble and mass-weighted average, the constitutive relations derived in various models still differ. The discrepancies between the various models appear in their formulation of the solid phase stress and the phase interaction. In some models, these terms were obtained from postulates.

There are very few models that provide separate energy equations for the fluid and solid phases. Those known to the authors include Ahmadi & Ma (1990) and Arnold *et al.* (1990). Because of the difference in their constitutive relations for the solid stress and the phase interaction in the momentum equations, the energy equations also differ. In fact, it is natural that any discrepancies in the momentum equations would propagate into the energy equations, because one utilizes the other in its derivation.

In their earlier work, Hwang & Shen (1989, 1991) established a two-phase model focusing on the derivation of the constitutive relations for the solid phase stress and the phase interaction in the momentum equations. These terms were derived from basic laws that apply at the particle level. In their study, the solid phase stress was expressed as

$$\{T^{\mathrm{s}}_{ij}\} = \{T^{\mathrm{c}}_{ij}\} + \{T^{\mathrm{k}}_{ij}\} + \{T^{\mathrm{p}}_{ij}\}, \qquad [5]$$

where $\{T^{\mathrm{c}}_{ij}\}$, $\{T^{\mathrm{k}}_{ij}\}$ and $\{T^{\mathrm{p}}_{ij}\}$ are the collisional stress, kinetic stress and particle presence stress, respectively. The collisional stress is caused by the momentum transfer from particle collisions. The kinetic stress, similar to the Reynolds stress in fluid turbulence, is due to the momentum transfer from the random motion of the particles. Details of the collisional and kinetic stress can be found in the granular flow literature. An excellent review is given by Campbell (1990). The particle presence stress is the portion of the solid stress resulting from the hydrodynamic stress on the particle's surface. The fluid phase stress was expressed as

$$\{T^{\mathrm{f}}_{ij}\} = \{T^{\mathrm{v}}_{ij}\} + \{T^{\mathrm{t}}_{ji}\}, \qquad [6]$$

where $\{T^{\mathrm{v}}_{ij}\}$ and $\{T^{\mathrm{t}}_{ij}\}$ are, the viscous and turbulence stress, respectively.

In the same study, the phase interaction term was derived as

$$\langle m_i \rangle = \frac{\langle c \rangle}{V_0}(\{f_i\} + \{f_{ai}\}) + \langle c \rangle \frac{\partial \{T^{\mathrm{f}}_{ji}\}}{\partial x_j} - \frac{\partial}{\partial x_j}(\langle c \rangle \{T^{\mathrm{p}}_{ji}\}), \qquad [7]$$

where V_0 is the volume of the particle; f_i is the drag force acting on a particle and f_{ai} represents the total of additional forces from the added mass, Basset force and Saffman force. The above expression to valid whenever the length scale of $\nabla\langle c \rangle$ is much larger than the particle's radius, R.

The above work is unique in its consistent application of a control volume approach to derive both the governing equations and the constitutive relations. In the present study, the same approach is applied to derive the fluctuation energy equations, thus completing the set of governing equations for turbulent fluid–solid flows.

With the inclusion of the constitutive relations, the system of balance equations is herein named the "framework". This framework is valid for all flow regimes. However, closure of the constitutive relations will depend on the different flow regimes. An example of such closure for a dilute turbulent flow of small particle Reynolds number can be found in Hwang (1989).

2. FLUCTUATION ENERGY EQUATIONS

A condensed derivation of the fluid-phase fluctuation energy equation is given below. Details can be found in Hwang (1989). The procedure to obtain the corresponding part for the solid phase is similar and thus omitted for brevity.

Consider a control volume V sitting in a fluid–solid mixture, as shown in figure 1. The continuity and the momentum equations for the two phases, [1]–[4], are obtained by balancing the mass and linear momentum within the control volume, followed by an ensemble average. Similarly, balancing the total energy of the fluid within the control volume, one can obtain the following equation for a single realization:

$$\frac{\partial}{\partial t}\int_V \rho_f(1-c)\left(\frac{v^2}{2}+e\right)\mathrm{d}V+\int_S \rho_f(1-c)\left(\frac{v^2}{2}+e\right)v_i\hat{\mathbf{N}}_i\,\mathrm{d}S=\int_S(1-c)\hat{\mathbf{N}}_i T^{\mathrm{v}}_{ij}v_j\,\mathrm{d}S$$
$$+\int_V \rho_f(1-c)g_i v_i\,\mathrm{d}V+\int_V m^{\mathrm{f}}_i v_i\,\mathrm{d}V-\int_{S_\mathrm{f}} q^{\mathrm{e}}_i\hat{\mathbf{N}}_i\,\mathrm{d}S_\mathrm{f}-\int_{S_\mathrm{i}} q^{\mathrm{i}}_i\check{\mathbf{n}}_i\,\mathrm{d}S_\mathrm{i}+\int_\mathrm{V}(1-\mathrm{c})\gamma^{\mathrm{f}}\,\mathrm{d}V. \quad [8]$$

Here e denotes the thermal energy per unit mass of the fluid as defined in thermodynamics; $m^{\mathrm{f}}_i=-m_i$ is the phase interaction force acting on the fluid by the solid per unit volume of the mixture; q^{e}_i and q^{i}_i denote, respectively, the heat flux through surfaces S_f and S_i; $\hat{\mathbf{N}}_i$ and $\check{\mathbf{n}}_i$ are the unit normal vectors shown in figure 1; and γ^{f} represents a distributed heat source per unit volume of the fluid, such as the radiation heat.

Ensemble averaging [8], followed by application of Gauss's theorem to surface integral terms and then removal of the integral sign, results in the following ensemble-averaged equation:

$$\frac{\partial}{\partial t}\left\langle\rho^{\mathrm{f}}\left(\frac{v^2}{2}+e\right)\right\rangle+\frac{\partial}{\partial x_i}\left\langle\rho^{\mathrm{f}}\left(\frac{v^2}{2}+e\right)v_i\right\rangle=\frac{\partial}{\partial x_i}\langle(1-c)T^{\mathrm{v}}_{ij}v_j\rangle$$
$$+\langle\rho^{\mathrm{f}}g_i v_i\rangle+\langle m^{\mathrm{f}}_i v_i\rangle-\left\langle(1-c)\frac{\partial q_i}{\partial x_i}\right\rangle+\langle(1-c)\gamma^{\mathrm{f}}\rangle. \quad [9]$$

Here q_i denotes the total heat flux leaving the fluid phase. Substituting $v_i=\{v_i\}+v''_i$ into the above equation yields the total energy equation for the fluid phase:

$$\langle\rho^{\mathrm{f}}\rangle\left(\frac{\partial}{\partial t}\left(\frac{\{v_i\}\{v_i\}}{2}+\left\{\frac{v''_i v''_i}{2}\right\}+\{e\}\right)+\{v_j\}\frac{\partial}{\partial x_j}\left(\frac{\{v_i\}\{v_i\}}{2}+\left\{\frac{v''_i v''_i}{2}\right\}+\{e\}\right)\right)$$
$$=\frac{\partial}{\partial x_j}(\langle 1-c\rangle\{T^{\mathrm{v}}_{ji}\}\{v_i\})+\frac{\partial}{\partial x_j}(\langle 1-c\rangle\{T^{\mathrm{v}}_{ji}v''_i\})-\frac{\partial}{\partial x_j}(\langle\rho^{\mathrm{f}}\rangle\{v''_i v''_j\}\{v_i\})-\frac{\partial}{\partial x_j}\left(\langle\rho^{\mathrm{f}}\rangle\left\{\frac{v''_i v''_i}{2}v''_j\right\}\right)$$
$$+\langle\rho^{\mathrm{f}}\rangle g\{v_i\}+\langle m^{\mathrm{f}}_i\rangle\{v_i\}+\langle m^{\mathrm{f}}_i v''_i\rangle-\frac{\partial}{\partial x_j}(\langle\rho^{\mathrm{f}}\rangle\{ev''_j\})-\langle 1-c\rangle\left\{\frac{\partial q_i}{\partial x_i}\right\}+\langle 1-c\rangle\{\gamma^{\mathrm{f}}\}, \quad [10]$$

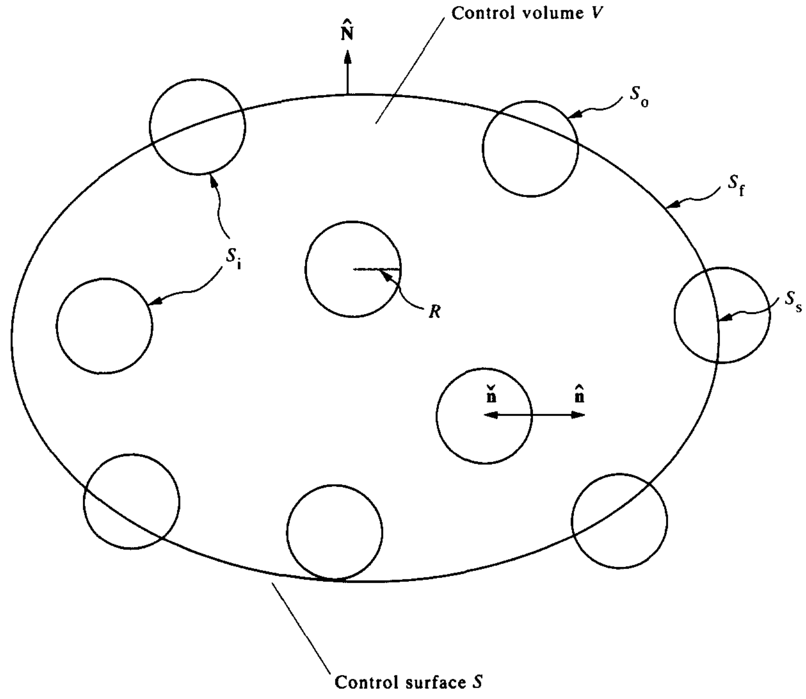

Figure 1. Arbitrary control volume set in the mixture.

where v_i'' denotes the fluid fluctuation velocity. The averaged quantity $\{v_i''v_i''/2\}$ represents the turbulence energy of the fluid. The term $\partial/\partial x_j(\langle\rho^{\mathrm{f}}\rangle\{ev_j''\})$ on the right-hand side of the above equation is the heat transfer due to turbulence.

The mechanical energy equation governing the energy of the mean fluid motion can be obtained by multiplying the fluid momentum equation [4] by the averaged fluid velocity as follows:

$$\langle\rho^{\mathrm{f}}\rangle\left(\frac{\partial\{v_i\}}{\partial t}+\{v_j\}\frac{\partial\{v_i\}}{\partial x_j}\right)\{v_i\}=\{v_i\}\frac{\partial}{\partial x_j}(\langle 1-c\rangle\{T_{ji}^{\mathrm{f}}\})+\langle\rho^{\mathrm{f}}\rangle g_i\{v_i\}-\langle m_i\rangle\{v_i\}. \qquad [11]$$

In the above, $\{T_{ji}^{\mathrm{f}}\}$ includes the averaged viscous stress $\{T_{ji}^{\mathrm{v}}\}$ and the averaged turbulence stress—$\{\rho_{\mathrm{f}}\mathrm{v}_i''\mathrm{v}_j''\}$. Hence,

$$\langle\rho^{\mathrm{f}}\rangle\left(\frac{\partial}{\partial t}\left(\frac{\{v_i\}\{v_i\}}{2}\right)+\{v_j\}\frac{\partial}{\partial x_j}\left(\frac{\{v_i\}\{v_i\}}{2}\right)\right)=\{v_i\}\frac{\partial}{\partial x_j}(\langle 1-c\rangle\{T_{ji}^{\mathrm{v}}\})$$
$$-\{v_i\}\frac{\partial}{\partial x_j}(\langle\rho^{\mathrm{f}}\rangle\{v_i''v_j''\})+\langle\rho^{\mathrm{f}}\rangle g_i\{v_i\}+\langle m_i^{\mathrm{f}}\rangle\{v_i\}. \qquad [12]$$

The thermal energy equation for a pure fluid flow has been derived in fluid mechanics as

$$\rho_{\mathrm{f}}\left(\frac{\partial e}{\partial t}+v_i\frac{\partial e}{\partial x_i}\right)=T_{ji}^{\mathrm{v}}\frac{\partial v_1}{\partial x_j}-\frac{\partial q_1}{\partial x_i}+\gamma^{\mathrm{f}}. \qquad [13]$$

This equation describes the balancing of thermal energy in a given infinitesimal volume of fluid. Integrating all the thermal energy of the fluid phase inside the control volume for a single realization yields

$$\frac{\partial}{\partial t}\int_V\rho_{\mathrm{f}}(1-c)e\,\mathrm{d}V+\int_S\rho_{\mathrm{f}}(1-c)ev_i\hat{\mathbf{N}}_i\,\mathrm{d}S=\int_V(1-c)T_{ji}^{\mathrm{v}}\frac{\partial v_1}{\partial x_j}\,\mathrm{d}V$$
$$-\int_{S_{\mathrm{f}i}}q_i\hat{\mathbf{n}}_i^{\mathrm{f}}\,\mathrm{d}S_{\mathrm{f}i}+\int_V(1-c)\gamma^{\mathrm{f}}\,\mathrm{d}V, \qquad [14]$$

where $S_{\mathrm{f}i}=S_{\mathrm{f}}+S_{\mathrm{i}}$ and $\hat{\mathbf{n}}_i^{\mathrm{f}}$ is the unit normal vector of $S_{\mathrm{f}i}$ as shown in figure 1. After ensemble averaging of the above equation and applying Gauss's theorem in the derivation, the thermal equation for the first phase in a two-phase flow can be obtained:

$$\langle\rho^{\mathrm{f}}\rangle\left(\frac{\partial\{e\}}{\partial t}+\{v_i\}\frac{\partial\{e\}}{\partial x_i}\right)=\langle 1-c\rangle\left\{T_{ji}^{\mathrm{v}}\frac{\partial v_1}{\partial x_j}\right\}-\frac{\partial}{\partial x_i}(\langle\rho^{\mathrm{f}}\rangle\{ev_i''\})-\langle 1-c\rangle\left\{\frac{\partial q_1}{\partial x_j}\right\}+\langle(1-c)\rangle\{\gamma^{\mathrm{f}}\}. \qquad [15]$$

As in the pure fluid equations, the correlation $\{ev_i''\}$ governs the heat flux caused by the fluctuation of fluid parcels.

The fluctuation energy equation for the fluid phase can then be obtained by subtracting the mechanical energy equation [12] and the thermal energy equation [15] from the total energy equation [11] to yield:

$$\langle\rho^{\mathrm{f}}\rangle\left(\frac{\partial}{\partial t}\left\{\frac{v_i''v_i''}{2}\right\}+\{v_j\}\frac{\partial}{\partial x_j}\left\{\frac{v_i''v_i''}{2}\right\}\right)$$
$$=\langle 1-c\rangle\{T_{ji}^{\mathrm{v}}\}\frac{\partial\{v_i\}}{\partial x_j}-\langle 1-c\rangle\left\{T_{ji}^{\mathrm{v}}\frac{\partial v_i}{\partial x_j}\right\}-(\langle\rho^{\mathrm{f}}\rangle\{v_i''v_j''\})\frac{\partial\{v_i\}}{\partial x_j}$$
$$+\frac{\partial}{\partial x_j}(\langle 1-c\rangle\{T_{ji}^{\mathrm{v}}v_i''\})-\frac{\partial}{\partial x_j}\left(\langle\rho^{\mathrm{d}}\rangle\left\{\frac{v_i''v_i''}{2}v_j''\right\}\right)+\langle m_i^{\mathrm{f}}v_i''\rangle. \qquad [16]$$

The heat flux and heat source terms all disappear from the above equation. If we consider an incompressible fluid flow, i.e. $\{\partial v_i/\partial x_i\} = 0$, the final form of the fluctuation energy equation for the fluid phase can be simplified to yield:

$$\begin{aligned}\langle\rho^{\mathrm{f}}\rangle&\left(\frac{\partial}{\partial t}\left\{\frac{v_i''v_i''}{2}\right\}+\{v_j\}\frac{\partial}{\partial x_j}\left\{\frac{v_i''v_i''}{2}\right\}\right)\\ &=-\langle 1-c\rangle\{p^{\mathrm{f}}\}\frac{\partial\{v_i\}}{\partial x_i}-(\langle\rho^{\mathrm{f}}\rangle\{v_i''v_j''\})\frac{\partial\{v_i\}}{\partial x_j}-\frac{\partial}{\partial x_j}\left(\langle\rho^{\mathrm{f}}\rangle\left\{\left(\frac{v_i''v_i''}{2}+\frac{p^{\mathrm{f}''}}{\rho_{\mathrm{f}}}\right)v_j''\right\}\right)\\ &\quad+\mu\frac{\partial}{\partial x_j\partial x_j}\left(\langle 1-c\rangle\left\{\frac{v_i''v_i''}{2}\right\}\right)-\mu\langle 1-c\rangle\left\{\frac{\partial v_i''}{\partial x_j}\frac{\partial v_i''}{\partial x_j}\right\}+\langle m_{\mathrm{i}}^{\mathrm{f}}\,\mathrm{v}_{\mathrm{i}}''\rangle, \end{aligned} \qquad [17]$$

where $\{p^{\mathrm{f}}\}$ and $p^{\mathrm{f}''}$ are the mass-weighted average and the fluctuation part of fluid pressure, respectively.

Some of the terms in [17] also appear in the fluctuation energy equation of a pure fluid flow. For instance, the second to fifth terms on the right-hand side can be identified with the turbulence production, the turbulence diffusion, the viscous diffusion and the viscous dissipation term, respectively, according to the terminology for turbulent fluid flow. The first term, however, has not appeared explicitly in the literature of turbulent two-phase flows. This term suggests that the fluctuation energy may be increased or decreased depending on the convergence or divergence of the mean fluid motion. In a two-phase flow with an incompressible fluid phase, $\{\partial v_i/\partial x_i\}$ vanishes. However, $\partial\{v_i\}/\partial x_i$ may not. This term is caused by the unsteady and nonuniform solid concentration, as discussed in Hwang (1989).

The fluctuation energy equation for the solid phase can be obtained in a similar manner to yield:

$$\begin{aligned}\langle\rho^{\mathrm{s}}\rangle&\left(\frac{\partial}{\partial t}\left\{\frac{u_i''u_i''}{2}\right\}+\{u_j\}\frac{\partial}{\partial x_j}\left\{\frac{u_i''u_i''}{2}\right\}\right)\\ &=\langle c\rangle\{T_{ji}^{\mathrm{c}}\}\frac{\partial\{u_i\}}{\partial x_j}+\langle c\rangle\{T_{ji}^{\mathrm{p}}\}\frac{\partial\{u_i\}}{\partial x_j}-(\langle\rho^{\mathrm{s}}\rangle\{u_i''u_j''\})\frac{\partial\{u_i\}}{\partial x_j}+\frac{\partial}{\partial x_i}(\langle c\rangle\{T_{ij}^{\mathrm{c}}u_j''\})\\ &\quad+\frac{\partial}{\partial x_i}(\langle c\rangle\{T_{ji}^{\mathrm{p}}u_j''\})-\frac{\partial}{\partial x_j}\left(\langle\rho^{\mathrm{s}}\rangle\left\{\frac{u_i''u_i''}{2}u_j''\right\}\right)+\langle m_i u_i''\rangle-\langle c\rangle\{\gamma^{\mathrm{s}}\}, \end{aligned} \qquad [18]$$

where u_i'' is the fluctuation velocity of the solid phase; $\{u_i''u_i''/2\}$ is its fluctuation energy; and $\{\gamma^{\mathrm{s}}\}$ denotes the dissipation rate of solid fluctuation energy. This dissipation is caused by collisions between the inelastic particles, the heat thus generated contributes to the heat flux term q_i^{i} in [8].

Note that the phase interaction terms in the fluctuation energy equations [17] and [18] are expressed symbolically as $\langle m_i^{\mathrm{f}} v_i''\rangle$ and $\langle m_i u_i''\rangle$. The constitutive expressions for these terms are derived in the next section based on the same concept as used in Hwang & Shen (1991) to obtain the phase interaction in the momentum equations.

3. PHASE INTERACTION TERMS

The phase interaction terms in the fluctuation energy equations must be derived following the same procedures as applied to the rest of the terms in the same equations; namely, they should be the difference of the phase interaction terms in the total energy equations minus that in the mechanical energy equations. (Note that there is no m_i-related phase interaction in thermal energy equations.)

As shown in figure 1, for any single realization, the total work done by the fluid on the solid through the fluid–solid interface inside a control volume is $\int_{S_i}\hat{\mathbf{n}}_i\,\Sigma_{ij}\,u_j\,\mathrm{d}S_i$, in which Σ_{ij} denotes the stress on (or inside, when particles lie on the control surface) the particle's surface due to the hydrodynamic effect. Note that the work done by the hydrodynamic forces on the solid portion of the control surface S_{s} has been included in the term $\int_S c\hat{\mathbf{N}}_i T_{ij}^{\mathrm{p}}\,u_j\,\mathrm{d}S$ in the total energy equation for the solid, in which T_{ij}^{p} denotes the stress on the solid intersection S_{s} due to the hydrodynamic effect. The mass-weighted average on T_{ij}^{p} gives the particle presence stress $\{T_{ij}^{\mathrm{p}}\}$ in [5]. Thus, all energy transfer to the solid phase inside the control volume has been accounted for, and separated into the phase interaction, $\int_{S_i} \hat{\mathbf{n}}_i \Sigma_{ij} u_j \, \mathrm{d}S_i$, and surface flux, $\int_S c \hat{\mathbf{N}}_i T^{\mathrm{p}}_{ij} u_j \, \mathrm{d}S$, terms. This separation is exactly parallel to the development of the phase interaction term and the particle presence stress in the momentum equations.

For a single realization, the phase interaction in the total energy equation can be obtained as follows:

$$\begin{aligned}
\int_{S_i} \hat{\mathbf{n}}_i \Sigma_{ij} u_j \, \mathrm{d}S_i &= \int_{S_{si}} \hat{\mathbf{n}}^{\mathrm{s}}_i \Sigma_{ij} u_j \, \mathrm{d}S_{si} - \int_{S_{\mathrm{s}}} \hat{\mathbf{N}}_i \Sigma_{ij} u_j \, \mathrm{d}S_{\mathrm{s}} \\
&= \int_{V_{\mathrm{s}}} \frac{\partial}{\partial x_i} (\Sigma_{ij} u_j \, \mathrm{d}V_{\mathrm{s}} - \int_S c \hat{\mathbf{N}}_i \Sigma_{ij} u_j \, \mathrm{d}S \\
&= \int_{V_{\mathrm{s}}} \left(\frac{\partial \Sigma_{ij}}{\partial x_i} \right) u_j \, \mathrm{d}V_{\mathrm{s}} + \int_{V_{\mathrm{s}}} \Sigma_{ij} \frac{\partial u_j}{\partial x_i} \, \mathrm{d}V_{\mathrm{s}} - \int_S c \hat{\mathbf{N}}_i \Sigma_{ij} u_j \, \mathrm{d}S \\
&= \int_V c \hat{\mathbf{h}}_i u_i \, \mathrm{d}V - \int_S c \hat{\mathbf{N}}_i T^{\mathrm{p}}_{ij} u_j \, \mathrm{d}S,
\end{aligned} \qquad [19]$$

where $S_{si} = S_{\mathrm{s}} + S_i$, $\hat{\mathbf{n}}^{\mathrm{s}}_i$ denotes the unit normal vector of S_{si} pointing outwards from the solid; V_{s} is the solid volume inside the control volume; and $\hat{\mathbf{h}}_i = \partial \Sigma_{ji} / \partial x_j$ is, physically, the resultant surface force acting on an infinitesimal volume of the solid caused by the hydrodynamic effect of the surrounding fluid. The term $\int_{V_{\mathrm{s}}} \Sigma_{ij} \partial u_j / \partial x_i \, \mathrm{d}V_{\mathrm{s}}$ vanishes because

$$\begin{aligned}
\int_{V_{\mathrm{s}}} \Sigma_{ij} \frac{\partial u_j}{\partial x_i} \, \mathrm{d}V_{\mathrm{s}} &= \int_{V_s} \Sigma_{ij} (e^{\mathrm{s}}_{ij} + \Omega^{\mathrm{s}}_{ij}) \, \mathrm{d}V_{\mathrm{s}} \\
&= \int_{V_{\mathrm{s}}} \Sigma_{ij} e^{\mathrm{s}}_{ij} + \Sigma_{ij} \Omega^{\mathrm{s}}_{ij} \, \mathrm{d}V_{\mathrm{s}} \\
&= 0,
\end{aligned}$$

where e^{s}_{ij} is the rate of deformation of the solid phase, which is zero from nondeformable materials. The last term in the above equation equals zero becuase it is the product of a symmetric tensor and an antisymmetric tensor.

The ensemble average of [19] gives

$$\left\langle \int_{S_i} \hat{\mathbf{n}}_i \Sigma_{ij} u_j \, \mathrm{d}S_i \right\rangle = \int_V \langle c \hat{\mathbf{h}}_i u_i \rangle \, \mathrm{d}V - \int_S \hat{\mathbf{N}}_i \langle c T^{\mathrm{p}}_{ij} u_j \rangle \, \mathrm{d}S. \qquad [20]$$

Applying Gauss's theorem and removing the integral sign yields the phase interaction term in the total energy equation for the solid phase:

$$\langle m_i u_i \rangle = \langle c \hat{\mathbf{h}}_i u_i \rangle - \frac{\partial}{\partial x_i} \langle c T^{\mathrm{p}}_{ij} u_j \rangle. \qquad [21]$$

Using the expression of $\langle m_i \rangle$ in Hwang & Shen (1991), the phase interaction term in the mechanical energy equation for the solid phase can be expressed as

$$\langle m_i \rangle \{u_i\} = \langle c \hat{\mathbf{h}}_i \rangle \{u_i\} - \{u_i\} \frac{\partial}{\partial x_j} \langle c T^{\mathrm{p}}_{ji} \rangle. \qquad [22]$$

The phase interaction term in the fluctuation energy equation of the solid can then be obtained by subtracting [22] from [21] as shown below:

$$\begin{aligned}
\langle m_i u_i'' \rangle &= \langle m_i u_i \rangle - \langle m_i \rangle \{u_i\} \\
&= \langle c \hat{\mathbf{h}}_i u_i'' \rangle - \frac{\partial}{\partial x_i} \langle c T^{\mathrm{p}}_{ij} u_j'' \rangle - \langle c T^{\mathrm{p}}_{ji} \rangle \frac{\partial \{u_j\}}{\partial x_i}.
\end{aligned} \qquad [23]$$

The first term on the right-hand side can be further reduced to yield:

$$\langle m_i u_i'' \rangle = \frac{\langle c \rangle}{V_0} (\{f_i u_i''\} + \{f_{ai} u_i''\}) - \frac{\partial}{\partial x_i} \langle c \rangle \{T^{\mathrm{p}}_{ij} u_j''\} - \langle c \rangle \{T^{\mathrm{p}}_{ij}\} \frac{\partial \{u_j\}}{\partial x_i}. \qquad [24]$$

Substituting this expression into [18], it is seen that terms involving T^p_{ij} drop out of the solid-phase fluctuation energy equation. This is analogous to what was found in the solid momentum equation (Hwang & Shen 1991). The T^p_{ij} effect, however, is present in the fluid-phase fluctuation equation through the phase interaction $\langle m^f_i v''_i \rangle$.

Owing to the no-slip condition, the velocities of the solid and the fluid at any point of the fluid–solid interface are identical. The local traction forces for the solid ($\hat{\mathbf{n}}_i \Sigma_{ij}$ in [18]) and the fluid are equal in magnitude and opposite in sense. Therefore, the phase interaction term in the total energy equation for the fluid phase satisfies

$$\langle m^f_i v_i \rangle = -\langle m_i u_i \rangle. \qquad [25]$$

The phase interaction term in the fluctuation energy equation for the fluid phase can then be determined in terms of $\langle m_i u''_i \rangle$:

$$\begin{aligned} \langle m^f_i v''_i \rangle &= \langle m^f_i v_i \rangle - \langle m^f_i \rangle \{v_i\} \\ &= -\langle m_i u_i \rangle + \langle m_i \rangle \{v_i\} \\ &= -\langle m_i u''_i \rangle - \langle m_i \rangle \{u_i\} + \langle m_i \rangle \{v_i\} \\ &= -\langle m_i u''_i \rangle + \langle m_i \rangle (\{v_i\} - \{u_i\}). \end{aligned} \qquad [26]$$

It is worthwhile noting that the phase interaction terms of the solid and fluid in the fluctuation energy equations are not numerically equal with an opposite sign, but differ by an amount $\langle m_i \rangle (\{v_i\} - \{u_i\})$. This quantity is exactly the difference between the phase interaction terms in the mechanical energy equations. Thus, interfacial force does mechanical work. This flow of mechanical energy is, however, not completely transferred from one phase to the other. The difference in the phase interaction term in the solid and fluid fluctuation energy equations has not been formulated this way in the existing literature. When the drag force dominates the phase interaction in the momentum equation, i.e. $\langle m_i \rangle$, the term $\langle m_i \rangle (\{v_i\} - \{u_i\})$ is easily shown to be positive, which means an enhancement of the fluid turbulence. One can postulate from physical reasons that this enhancement is from the wake effect caused by the presence of particles.

4. DISCUSSION

With the present work, the mathematical framework for two-phase flows has (Hwang & Shen 1989, 1991) been extended to balancing the fluctuation energy. The application of this framework requires detailed modeling or the closure process.

The objective of the detailed modeling is to express, physically and mathematically, each term within the framework in terms of the averaged quantities, e.g. $\{v_i\}$, $\{u_i\}$, $\langle c \rangle$, $\{p^f\}$, $\{v''_i v''_i\}/2$ or $\{u''_i u''_i\}/2$. These dependent variables can then be determined by solving the system of equations. Boundary conditions, in general, are also required. To achieve this objective, the knowledge of granular flow theories, the turbulence models of a single-phase fluid and the modeling of $\{u''v''\}$ are essential, among others. Although not included herein, details of a proposed model for turbulent flow of a dilute mixture are given in Hwang (1989).

In brief, closure of the governing equations can be summarized as follows. No additional equation is needed for the modeling of the solid-phase portion; because the appropriate modeling of each term is directly obtainable from granular flow theories (Shen & Ackermann 1982; Lun *et al.* 1984; Jenkins & Richman 1985; Babič 1985). Additional equation(s) may be desired for the fluid phase, depending on the chosen turbulence model. For instance, if a closure of k–ϵ models is chosen, the balance equation for the dissipation of fluctuation fluid energy, i.e. the ϵ-equation, should be added. In this case, the fluctuation energy equation for the fluid phase, [17], serves as the k-equation. Since the k–ϵ models for a pure fluid are most preferable in industrial use (Launder & Spalding 1972, 1974), they are recommended to be used in the present framework for problem-solving. Existing two-phase models that apply k–ϵ turbulence models are those of Elghobashi & Abou-Arab (1983). Chen & Wood (1985) and Ma (1987); although the frameworks developed by them are not the same as the present one.

It should be noted that the k–ϵ models (two-equation models) are not the only option for detailed modeling, other models, e.g. zero-equation models (mixing-length models), one-equation models or multi-equation turbulence models, can also be applied depending on the nature of the problem and the users' judgment. While the turbulence theory for a single-phase fluid is not yet conclusive, a similar situation is faced by two-phase modeling. The framework derived in this paper, however, is independent of the turbulence model used.

5. CONCLUSIONS

The fluctuation energy equations [17] and [18] and the associated constitutive relations [24] and [26] have been derived on the same concept as used to derive the mass and momentum equations (Hwang & Shen 1989, 1991). A consistent control-volume approach has been used to obtain both the governing equations and the constitutive relations. The results differ from other two-phase models mainly in the formulations of the terms related to the solid phase stress and the phase interaction.

It is found that the phase interaction terms in the fluctuation energy equations for the fluid and the solid are not numerically equal with an opposite sign, but differ by a term $\langle m_i \rangle (\{v_i\} - \{u_i\})$. This term enhances the fluid turbulence when the hydrodynamic interaction is dominated by the drag force. Together with the mass and momentum equations given in Hwang & Shen (1989, 1991), the present work provides a complete framework for modeling turbulent two-phase flows, in which the fluid is incompressible and the solid is rigid. The only assumption in this work is that the length scale $\nabla \langle c \rangle$ is much larger than the particle radius. This framework can serve as the basis for detailed modeling of specific flow regimes.

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